

Labeled Packing of Non Star Tree into its Fifth Power and Sixth Power

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Abstract

In this paper we prove that we can find a labeled packing of a non star tree T into T^6 with $m_T + \lceil \frac{n-m_T}{5} \rceil$ labels, where n is the number of vertices of T and m_T is the maximum number of leaves that can be removed from T in such a way that the obtained graph is a non star tree. Also, we prove that we can find a labeled packing of a non star tree T into T^5 with $m_T + 1$ labels and a labeled packing of a path P_n , $n \geq 4$, into P_n^4 with $\lceil \frac{n}{4} \rceil$ labels.

1 Introduction

All graphs considered in this paper are finite and undirected. For a graph G , $V(G)$ and $E(G)$ will denote its vertex set and edge set, respectively. We denote by $N_G(x)$ the set of the neighbors of the vertex x in G . The degree $d_G(x)$ of the vertex x in G is the cardinality of the set $N_G(x)$. For short, we use $d(x)$ instead of $d_G(x)$ and $N(x)$ instead of $N_G(x)$. The distance between two vertices of G , say x and y , is denoted by $dist_G(x, y)$, and for short we usually use $dist(x, y)$. For a subset U of V , we denote by $G - U$ the graph obtained from G by deleting all the vertices in $U \cap V$ and their incident edges. For a subset F of E , we write $G - F := (V, E \setminus F)$.

A vertex of degree one in a tree T is called a leaf and the neighbor of a leaf is its father. For a non star tree T , we denote by m_T the maximum number of leaves that can be removed from T in such a way that the obtained tree is a non star one. The number of edges of a path P is its length $l(P)$. A path on n vertices is denoted by P_n . The middle vertex of P_5 will be called a bad vertex.

Let G be a graph of order n . Consider a permutation $\sigma : V(G) \rightarrow V(K_n)$, the map $\sigma^* : E(G) \rightarrow E(K_n)$ such that $\sigma^*(xy) = \sigma(x)\sigma(y)$ is the map induced by σ . We say that there is a packing of k copies of G (into the complete graph K_n) if there exist permutations $\sigma_i : V(G) \rightarrow V(K_n)$, where $i = 1, \dots, k$, such that $\sigma_i^*(E(G)) \cap \sigma_j^*(E(G)) = \emptyset$ for $i \neq j$. A packing of k copies of a graph G will be called a k -placement of G . A packing of two copies of G (i.e. a 2-placement) is also called an embedding of G (into its complement \bar{G}). That is, we say that G can be embedded in its complement if there exists a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. A permutation σ on $V(G)$ such that $\sigma(x) \neq x$ for every x in $V(G)$ is called a fixed point free permutation.

The problem of embedding paths and trees in their complements has long been one of the fundamental questions in combinatorics that has been considerably investigated [2,4,5,6,7,8]. For recent results and survey on this field, we refer to the survey papers of Wozniak [9] and Yap [10].

Concerning non star trees, the following theorem was proved by Straight (unpublished, cf. [3])

Theorem 1.1 *Let T be a non star tree, then T is contained in its own complement.*

This result has been improved in many ways especially in considering some additional information and conditions about embedding. An example of such a result is the following theorem contained as a lemma in [9]:

Theorem 1.2 *Let T be a non-star tree of order n with $n > 3$. Then there exists a 2-placement σ of T such that for every $x \in V(T)$, $\text{dist}(x, \sigma(x)) \leq 3$.*

This theorem immediately implies the following:

Corollary 1.1 *Let T be a non-star tree of order n with $n > 3$. Then there exists an embedding of T such that $\sigma(T) \subseteq T^7$.*

In [6], Kheddouci *et al.* gave a better improvement in the following theorem:

Theorem 1.3 *Let T be a non star tree and let x be a vertex of T . Then, there exists a permutation σ on $V(T)$ satisfying the following four conditions:*

1. σ is a 2-placement of T .
2. $\sigma(T) \subseteq T^4$.
3. $\text{dist}(x, \sigma(x)) = 1$.
4. for every neighbor y of x , $\text{dist}(y, \sigma(y)) \leq 2$.

Labeled graph packing is a well known field of graph theory that has been considerably investigated. It is introduced by E. Duchene and H. Kheddouci. Below is the definition of the labeled packing problem:

Definition 1.1 . *Consider a graph G . Let f be a mapping from $V(G)$ to the set $\{1, 2, \dots, p\}$. The mapping f is called a p -labeled-packing of k copies of G into K_n if there exists permutations $\sigma_i : V(G) \rightarrow V(K_n)$, where $i = 1, \dots, k$, such that:*

1. $\sigma_i^*(E(G)) \cap \sigma_j^*(E(G)) = \emptyset$; for all $i \neq j$.
2. For every vertex v of G , we have $f(v) = f(\sigma_1(v)) = f(\sigma_2(v)) = \dots = f(\sigma_k(v))$.

The maximum positive integer p for which G admits a p -labeled-packing of k copies of G is called the labeled packing number of k copies of G and is denoted by $\lambda^k(G)$.

E. Duchene *et al.* introduced the following two results which are presented as Lemmas in [1]. These results give an upper bound for the labeled packing $\lambda^2(G)$.

Theorem 1.4 *Let G be a graph of order n and let I be a maximum independent set of G . If there exists an embedding of G into K_n , then*

$$\lambda^2(G) \leq |I| + \lfloor \frac{n-|I|}{2} \rfloor$$

Theorem 1.5 *Let G be a graph of order n with a maximum independent set I of size at least $\lfloor \frac{n}{2} \rfloor$. If there exists a packing of $k \geq 2$ copies of G into K_n , then*

$$\lambda^k(G) \leq |I| + \lfloor \frac{n-|I|}{k} \rfloor$$

E. Duchene *et al.* [1] introduced and studied the labeled graph packing problem for some vertex labeled graphs. In this paper, we are concerned with finding a p -labeled-packing of G into G^k . We give below the definition of the new labeled packing problem:

Definition 1.2 *Let f be a mapping from $V(G)$ to the set $\{1, 2, \dots, p\}$. The mapping f is called p -labeled-packing of G into G^k if there exists a permutation $\sigma : V(G) \rightarrow V(K_n)$, such that:*

1. σ is a 2-placement of G .
2. $\sigma(G) \subseteq G^k$.
3. For every vertex v of G , we have $f(v) = f(\sigma(v))$.

The maximum positive integer p for which G admits a p -labeled-packing of G into G^k is called labeled packing k -power number and denoted by $w^k(G)$.

Concerning the packing of a path P_n , $n \geq 4$, into P_n^4 , we introduce the following result:

Theorem 1.6 *Consider a path P_n , $n \geq 4$, and let u and v be its end vertices. Then there exists a permutation σ on $V(P_n)$ such that σ satisfies the following conditions:*

1. σ is a 2-placement.
2. $\sigma(P_n) \subseteq P_n^4$.
3. $\text{dist}(u, \sigma(u)) = 1$ and $\text{dist}(v, \sigma(v)) \leq 1$.
4. The length of each cycle of σ is at most 4.

This result allows us to establish the following:

Corollary 1.2 *Consider a path P_n , $n \geq 4$, then $w^4(P_n) \geq \lceil \frac{n}{4} \rceil$.*

To formulate our main results we need to introduce some definitions.

Let T be a non star tree and let x be a vertex of T . Then, a fixed point free permutation σ on $V(T)$ is called a (T, x) -well 2-placement if it satisfies the following conditions:

1. σ is a 2-placement of T .
2. $\sigma(T) \subseteq T^6$.
3. $\text{dist}(x, \sigma(x)) \leq 2$.
4. $\text{dist}(y, \sigma(y)) \leq 3$ for every neighbor y of x .
5. $\text{dist}(y, \sigma(y)) \leq 4$ for every y such that $d(y) = 1$.
6. The length of each cycle of σ is at most 5.

We prove first:

Theorem 1.7 *Let T be a non star tree and let x be a vertex of T . Then, there exists a (T, x) -well 2-placement.*

This implies the following:

Corollary 1.3 *Consider a non star tree T with $|V(T)| = n$, then $w^6(T) \geq m_T + \lceil \frac{n-m_T}{5} \rceil$.*

Let T be a non star tree and let x be a vertex of T . Then, a fixed point free permutation σ on $V(T)$ is called a (T, x) -good 2-placement if it satisfies the following conditions:

1. σ is a 2-placement of T .

2. $\sigma(T) \subseteq T^5$.
3. $\text{dist}(x, \sigma(x)) = 1$.
4. $\text{dist}(y, \sigma(y)) \leq 2$ for every neighbor y of x .
5. $\text{dist}(y, \sigma(y)) \leq 4$ for every y such that $d(y) = 1$.

We prove that:

Theorem 1.8 *Let T be a non star tree and let x be a vertex of T such that x is not a bad vertex. Then, there exists a (T, x) -good 2-placement.*

This result allows us to establish the following:

Corollary 1.4 *Consider a non star tree T , then $w^5(T) \geq m_T + 1$.*

2 Labeled Packing of P_n into P_n^4

In this section, we are going to prove Theorem 1.6, but we need first to prove the theorem for $n = 4, \dots, 7$:

Lemma 2.1 *Consider a path P_n such that $4 \leq n \leq 7$, and let u and v be its end vertices. Then there exists a permutation σ on $V(P_n)$ satisfying the following conditions:*

1. σ is a 2-placement.
2. $\sigma(P_n) \subseteq P_n^4$.
3. $\text{dist}(u, \sigma(u)) = 1$ and $\text{dist}(v, \sigma(v)) \leq 1$.
4. The length of each cycle of σ is at most 4.

Proof. For each path P_n , $n = 4, \dots, 7$, we will introduce below a permutation σ on $V(P_n)$, satisfying the above conditions:

For $P = x_1x_2x_3x_4$, $\sigma = (x_1 \ x_2 \ x_4 \ x_3)$.

For $P = x_1x_2x_3x_4x_5$, $\sigma = (x_1 \ x_2 \ x_5 \ x_4)(x_3)$.

For $P = x_1x_2x_3x_4x_5x_6$, $\sigma = (x_1 \ x_2 \ x_5 \ x_4)(x_3)(x_6)$.

For $P = x_1x_2x_3x_4x_5x_6x_7$, $\sigma = (x_1 \ x_2 \ x_5)(x_3 \ x_7 \ x_6)(x_4)$.

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Proof of Theorem 1.4.

The proof is by induction. By the previous Lemma, σ exists for $n = 4, \dots, 7$. Suppose now that $n \geq 8$ and the theorem holds for all $n' < n$. Then P_n can be partitioned into two paths P' and P'' such that $l(P'), l(P'') \geq 3$. Let x be the end vertex of P' and y that of P'' such that $d_{P_n}(x) = d_{P_n}(y) = 2$ and let σ' and σ'' be two permutations defined on $V(P')$ and $V(P'')$ respectively such that σ' and σ'' satisfy the four conditions mentioned in the theorem with $\text{dist}(x, \sigma'(x)) = 1$ and $\text{dist}(y, \sigma''(y)) \leq 1$. Let σ be a permutation defined on $V(P_n)$, such that:

$$\sigma(v) = \begin{cases} \sigma'(v) & \text{if } v \in V(P') \\ \sigma''(v) & \text{if } v \in V(P'') \end{cases}$$

It can be easily shown that σ satisfies the four conditions. \square

Proof of Corollary 1.2.

Consider a path P_n , $n \geq 4$, and let u and v be its end vertices. Then, by the previous theorem there exists a permutation σ on $V(P_n)$ satisfying the following conditions:

1. σ is a 2-placement.
2. $\sigma(P_n) \subseteq P_n^4$.
3. $\text{dist}(u, \sigma(u)) = 1$ and $\text{dist}(v, \sigma(v)) \leq 1$.
4. The length of each cycle of σ is at most 4.

Let r be the number of cycles of σ and let $\sigma_1, \dots, \sigma_r$ be these cycles. Note that $r \geq \lceil \frac{n}{4} \rceil$. Label the vertices of σ_i by i for $i = 1, \dots, r$. Hence, we obtain a labeled packing of P_n into P_n^4 with r labels and so $w^4(P_n) \geq \lceil \frac{n}{4} \rceil$. \square

3 Labeled Packing of a Non Star Tree T into T^5 and T^6

In this section, we are going to prove the main results of this paper, but we still need to introduce some definitions and results on paths followed by a sequence of lemmas.

Consider a path P_n , $n \geq 4$, and let x be a vertex of P_n . A fixed point free permutation σ on $V(P_n)$ is called a (P_n, x) -well path 2-placement, if it satisfies the following conditions:

1. σ is a 2-placement of P_n .
2. $\sigma(P_n) \subseteq P_n^6$.
3. $\text{dist}(x, \sigma(x)) \leq 2$.
4. $\text{dist}(y, \sigma(y)) \leq 3$ for every $y \in N(x)$ and for every y such that $d(y) = 1$.
5. The length of each cycle of σ is at most 5.

We will prove the following theorem:

Theorem 3.1 *Consider a path P_n and let x be a vertex of P_n , $n \geq 4$. Then there exists a (P_n, x) -well path 2-placement.*

Lemma 3.1 *Consider a path P_n , $4 \leq n \leq 7$, and let x be a vertex of P_n . Then there exists a (P_n, x) -well path 2-placement, say σ , such that $\text{dist}(v, \sigma(v)) \leq 3$ for every $v \in V(P_n)$.*

Proof. For each path P_n , $n = 4, \dots, 7$, and for every vertex x of P_n we will introduce below a (P_n, x) -well path 2-placement σ such that $\text{dist}(v, \sigma(v)) \leq 3$ for every $v \in V(P_n)$:

For $P = x_1x_2x_3x_4$, $\sigma = (x_1 \ x_2 \ x_4 \ x_3)$ is a (P, x) -well path 2-placement for every $x \in V(P)$.
 For $P = x_1x_2x_3x_4x_5$, $\sigma = (x_1 \ x_2 \ x_4 \ x_5 \ x_3)$ is a (P, x) -well path 2-placement for every $x \in V(P)$.
 For $P = x_1x_2x_3x_4x_5x_6$, there are three choices for choosing x , either x_1 , x_2 or x_3 . $\sigma = (x_1 \ x_2 \ x_4)(x_3 \ x_6 \ x_5)$ is a (P, x) -well path 2-placement for $x \in \{x_1, x_2\}$, and $\sigma = (x_3 \ x_1)(x_5 \ x_2)(x_6 \ x_4)$ is a (P, x_3) -well path 2-placement.
 For $P = x_1x_2x_3x_4x_5x_6x_7$, there are four choices for choosing x , either x_1 , x_6 , x_3 or x_4 . $\sigma = (x_1 \ x_2 \ x_5 \ x_3)(x_4 \ x_6 \ x_7)$ is a (P, x) -well path 2-placement for every x of the previous choices. \blacksquare

Proof of Theorem 3.1.

The proof is by induction. By the previous Lemma, there exists a (P_n, x) -well path 2-placement for every vertex x of P_n , where $4 \leq n \leq 7$. Suppose now $n \geq 8$ and the theorem holds for all $n' < n$. Let x be a vertex of P_n . Since $n \geq 8$, then P_n can be partitioned into two paths P' and P'' such that $l(P'), l(P'') \geq 3$. Without loss of generality, suppose that $x \in V(P')$. Let x_1 be the end vertex of P'' such that $d_{P_n}(x_1) = 2$. By induction, there exists a (P', x) -well path 2-placement, say σ_x , and there exists a (P'', x_1) -well path 2-placement, say σ_{x_1} . Let σ be a permutation defined on $V(P_n)$ such that:

$$\sigma(v) = \begin{cases} \sigma_x(v) & \text{if } v \in V(P') \\ \sigma_{x_1}(v) & \text{if } v \in V(P'') \end{cases}$$

It can be easily shown that σ is a (P_n, x) -well path 2-placement. \square

Consider a path P_n , $n \geq 4$, and let x be a vertex of P_n . We say that a fixed point free permutation σ on $V(P_n)$ is a (P_n, x) -good path 2-placement if σ satisfies the following conditions:

1. σ is a 2-placement of P_n .
2. $\sigma(P_n) \subseteq P_n^5$.
3. $\text{dist}(x, \sigma(x)) = 1$.
4. $\text{dist}(y, \sigma(y)) \leq 2$ for every $y \in N(x)$ and for every y such that $d(y) = 1$.

We prove:

Theorem 3.2 *Let x be a vertex of P_n , $n \geq 4$, such that x is not a bad vertex, then there exists a (P_n, x) -good path 2-placement.*

Lemma 3.2 *For every x in $V(P_n)$, $4 \leq n \leq 7$, such that x is not a bad vertex, there exists a (P_n, x) -good path 2-placement.*

Proof. For every x in $V(P_n)$, $n = 4, \dots, 7$, we will introduce below a (P_n, x) -good path 2-placement σ_x :

For $P = x_1x_2x_3x_4$, there are two choices for choosing x , either x_1 or x_2 . Then:

$$\sigma_{x_1} = (x_1 \ x_2 \ x_4 \ x_3); \sigma_{x_2} = (x_1 \ x_3 \ x_4 \ x_2)$$

For $P = x_1x_2x_3x_4x_5$, there are two choices for choosing x , either x_1 or x_4 . Then:

$$\sigma_{x_1} = \sigma_{x_4} = (x_1 \ x_2 \ x_4 \ x_5 \ x_3).$$

For $P = x_1x_2x_3x_4x_5x_6$, there are three choices for choosing x , either x_1 , x_2 or x_3 . Then:

$$\sigma_{x_1} = (x_1 \ x_2 \ x_4 \ x_3 \ x_6 \ x_5); \sigma_{x_2} = \sigma_{x_3} = (x_1 \ x_3 \ x_4 \ x_6 \ x_5 \ x_2).$$

For $P = x_1x_2x_3x_4x_5x_6x_7$, there are four choices for choosing x , either x_1 , x_2 , x_3 or x_4 . Then:

$$\sigma_{x_1} = \sigma_{x_4} = (x_1 \ x_2 \ x_4 \ x_5 \ x_7 \ x_6 \ x_3); \sigma_{x_2} = \sigma_{x_3} = (x_1 \ x_3 \ x_4 \ x_6 \ x_7 \ x_5 \ x_2).$$

■

Proof of Theorem 3.2.

The proof is by induction. By the previous Lemma, there exists a (P_n, x) -good path 2-placement for every $x \in V(P_n)$ such that x is not a bad vertex, where $4 \leq n \leq 7$. Suppose now $n \geq 8$ and the theorem holds for all $n' < n$. Let x be a vertex of P_n . Since $n \geq 8$, then P_n can be partitioned into two paths P' and P'' such that $l(P'), l(P'') \geq 3$. Without loss of generality, suppose that $x \in V(P')$ such that P' is chosen to be distinct from P_5 . Let x_1 be the end vertex of P'' such that $d_{P_n}(x_1) = 2$. By induction, there exists a (P', x) -good path 2-placement, say σ_x , and there exists a (P'', x_1) -good path 2-placement, say σ_{x_1} . Let σ be a permutation defined on $V(P_n)$ such that:

$$\sigma(v) = \begin{cases} \sigma_x(v) & \text{if } v \in V(P') \\ \sigma_{x_1}(v) & \text{if } v \in V(P'') \end{cases}$$

It can be easily shown that σ is a (P_n, x) -good path 2-placement. \square

Let T be a non star tree and let xy be an edge in T . We denote by $T_{(x,y)}$ the connected component containing x in $T - \{xy\}$ and it is called a neighbor tree of y . $T_{(x,y)}$ is said to be a neighbor F -tree of y if $T_{(x,y)}$ is a path of length at most two such that whenever $T_{(x,y)}$ is a path of length two, then x is an end vertex in it.

Lemma 3.3 *Consider a non star tree T containing a vertex x such that $d(x) > 2$. Let $\{x_1, \dots, x_n\}$, $n > 2$, be the neighbors of x . Suppose that $T_{(x_i,x)}$ is a neighbor F -tree of x for $i = 1, \dots, m$, where $2 \leq m < n$. Let T' be the connected component containing x in $T - \{xx_i; i = 1, \dots, m\}$ and let G be the graph obtained by the union of the remaining components. Suppose that there exists a (T', z) -well 2-placement σ such that $\text{dist}(x, \sigma(x)) \leq 3$, where z is a vertex in T' not necessarily distinct from x , then there exists a (T, z) -well 2-placement, say σ_z , such that $\sigma_z(v) = \sigma(v)$ for every $v \in V(T')$, $\text{dist}(x_i, \sigma_z(x_i)) \leq 2$ for $i = 1, \dots, m$ and $\text{dist}_T(\sigma_z(u), \sigma_z(v)) \leq 5$ whenever uv is an edge in G .*

Proof. Let r , p and q be the number of neighbor trees of x that are paths of length zero, one and two, respectively, in the set $\{T_{(x_i,x)}; i = 1, \dots, m\}$. In what follows we need to rename some neighbors of x for the sake of the proof. Let $T_i = T_{(x_i,x)}$ for $i = 1, \dots, m$ such that if $r > 0$, then T_i is the vertex a_i for $i = 1, \dots, r$, $T_i = b_{i-r}c_{i-r}$ for $i = r+1, \dots, r+p$ if $p > 0$ and $T_i = d_{i-(p+r)}e_{i-(p+r)}f_{i-(p+r)}$ for $i = r+p+1, \dots, r+p+q$ if $q > 0$. We will define a (T, z) -well 2-placement σ_z according to the different values of r , p and q . To construct σ_z , we need first to introduce the permutations Θ , Υ and Δ on $V(G)$ in each case below such that:

$$\Theta = \begin{cases} \theta & \text{if } r \text{ is even} \\ \theta' & \text{if } r \text{ is odd} \end{cases}, \Upsilon = \begin{cases} \epsilon & \text{if } p \text{ is even} \\ \epsilon' & \text{if } p \text{ is odd} \end{cases} \text{ and } \Delta = \begin{cases} \delta & \text{if } q \text{ is even} \\ \delta' & \text{if } q \text{ is odd} \end{cases}$$

where θ , θ' , ϵ , ϵ' , δ and δ' are permutations defined in each case below.

- Case 1. p , q and r are strictly greater than one.
 If $r = 2n'$ for some $n' \in \mathbb{Z}$, then let $\theta = \prod_{j=1}^{j=n'} (a_{2j-1} a_{2j})$. If $r = 2n' + 1$, then if $n' = 1$ let $\theta' = (a_1 a_2 a_3)$, else let $\theta' = (a_1 a_2 a_3) \prod_{j=2}^{j=n'} (a_{2j} a_{2j+1})$.
 If $p = 2m'$ for some $m' \in \mathbb{Z}$, then let $\epsilon = \prod_{j=1}^{j=m'} (b_{2j} c_{2j} b_{2j-1} c_{2j-1})$. If $p = 2m' + 1$, then let

$\epsilon' = (b_1 \ b_2 \ b_3)(c_1 \ c_3 \ c_2)$ if $m' = 1$, else let $\epsilon' = (b_1 \ b_2 \ b_3)(c_1 \ c_3 \ c_2) \prod_{j=2}^{j=m'} (b_{2j+1} \ c_{2j+1} \ b_{2j} \ c_{2j})$.
 If $q = 2s'$ for some $s' \in \mathbb{Z}$, then let $\delta = \prod_{j=1}^{j=s'} (e_{2j-1} \ f_{2j} \ d_{2j-1})(e_{2j} \ f_{2j-1} \ d_{2j})$. If $q = 2s' + 1$, then let $\delta' = (d_1 \ e_1 \ f_2 \ e_2 \ f_1)(d_2 \ d_3 \ f_3 \ e_3)$ if $s' = 1$, else let $\delta' = (d_1 \ e_1 \ f_2 \ e_2 \ f_1)(d_2 \ d_3 \ f_3 \ e_3) \prod_{j=2}^{j=s'} (e_{2j} \ f_{2j+1} \ d_{2j})(e_{2j+1} \ f_{2j} \ d_{2j+1})$.
 Finally, let $\sigma_z = \Theta \ \Upsilon \ \Delta \ \sigma$.

- Case 2. $r = 1$.

We will study the following subcases:

1. $p > 1$ and $q > 1$.

In this case, if $p = 2m'$ then let $\epsilon = (a_1 \ b_1)(b_2 \ c_2 \ c_1)$ if $m' = 1$, and if $m' > 1$ then let $\epsilon = (a_1 \ b_1)(b_2 \ c_2 \ c_1) \prod_{j=2}^{j=m'} (b_{2j} \ c_{2j} \ b_{2j-1} \ c_{2j-1})$. On the other hand, if $p = 2m' + 1$, then let $\epsilon' = (a_1 \ b_1 \ c_1) \prod_{j=1}^{j=m'} (b_{2j+1} \ c_{2j+1} \ b_{2j} \ c_{2j})$. Finally, let $\sigma_z = \Delta \ \Upsilon \ \sigma$, where δ and δ' are the same as in Case 1.

2. $p > 1$ and $q = 1$.

Let $\sigma_z = \Upsilon \ (a_1 \ d_1 \ f_1 \ e_1) \ \sigma$, where ϵ and ϵ' are the same as in Case 1.

3. $p > 1$ and $q = 0$.

Then let $\sigma_z = \Upsilon \ \sigma$, where ϵ and ϵ' are the same as in (1).

4. $p = 1$ and $q > 1$.

Then $\sigma_z = (a_1 \ b_1 \ c_1) \ \Delta \ \sigma$, where δ and δ' are the same as in Case 1.

5. $p = 1$ and $q = 1$.

Then let $\sigma_z = (a_1 \ b_1 \ c_1 \ e_1)(f_1 \ d_1) \ \sigma$.

6. $p = 1$ and $q = 0$.

Then $\sigma_z = (a_1 \ b_1 \ c_1) \ \sigma$.

7. $p = 0$ and $q > 1$.

If $q = 2s'$ then let $\delta = (d_1 \ e_1 \ f_2)(f_1 \ a_1 \ d_2 \ e_2)$ if $s' = 1$ and if $s' > 1$ then let $\delta = (d_1 \ e_1 \ f_2)(f_1 \ a_1 \ d_2 \ e_2) \prod_{j=2}^{j=s'} (e_{2j-1} \ f_{2j} \ d_{2j-1})(e_{2j} \ f_{2j-1} \ d_{2j})$. On the other hand, if $q = 2s' + 1$ then let $\delta' = (a_1 \ d_1 \ f_1 \ e_1) \prod_{j=1}^{j=s'} (e_{2j} \ f_{2j+1} \ d_{2j})(e_{2j+1} \ f_{2j} \ d_{2j+1})$. Finally, let $\sigma_z = \Delta \ \sigma$.

8. $p = 0$ and $q = 1$.

Then let $\sigma_z = (a_1 \ d_1 \ f_1 \ e_1) \ \sigma$.

- Case 3. $p = 1$.

1. $r > 1$ and $q > 1$.

If $r = 2n'$ then let $\theta = (a_1 \ b_1 \ c_1 \ a_2)$ if $n' = 1$ and if $n' > 1$ then let $\theta = (a_1 \ b_1 \ c_1 \ a_2) \prod_{j=2}^{j=n'} (a_{2j-1} \ a_{2j})$. On the other hand, if $r = 2n' + 1$ then let $\theta' = (b_1 \ c_1 \ a_1) \prod_{j=1}^{j=n'} (a_{2j} \ a_{2j+1})$. Let $\sigma_z = \Theta \ \Delta \ \sigma$, where δ and δ' are defined as in Case 1.

2. $r > 1$ and $q = 1$.

Then let $\sigma_z = (c_1 \ e_1 \ b_1)(f_1 \ d_1) \Theta \ \sigma$, where θ and θ' are the same as the ones defined in Case 1.

3. $r > 1$ and $q = 0$.

Then let $\sigma_z = \Theta \ \sigma$, where θ and θ' are just like the ones in (1).

4. $r = 0$ and $q > 1$.

If $q = 2s'$ then let $\delta = (d_1 \ f_1 \ b_1 \ c_1)(f_2 \ d_2)(e_1 \ e_2)$ if $s' = 1$ and if $s' > 1$ then let $\delta = (d_1 \ f_1 \ b_1 \ c_1)(f_2 \ d_2)(e_1 \ e_2) \prod_{j=2}^{j=s'} (e_{2j-1} \ f_{2j} \ d_{2j-1})(e_{2j} \ f_{2j-1} \ d_{2j})$. Otherwise, if $q = 2s' + 1$ then $\delta' = (c_1 \ e_1 \ f_1 \ d_1 \ b_1) \prod_{j=1}^{j=s'} (e_{2j} \ f_{2j+1} \ d_{2j})(e_{2j+1} \ f_{2j} \ d_{2j+1})$. Finally, let $\sigma_z = \Delta \sigma$.

5. $r = 0$ and $q = 1$.

Let $\sigma_z = (c_1 \ e_1 \ f_1 \ d_1 \ b_1) \sigma$.

- Case 4. $q = 1$.

1. $r > 1$ and $p > 1$.

If $r = 2n'$ then let $\theta = (d_1 \ f_1 \ e_1 \ a_1 \ a_2)$ if $n' = 1$ and if $n' > 1$ then let $\theta = (d_1 \ f_1 \ e_1 \ a_1 \ a_2) \prod_{j=2}^{j=n'} (a_{2j-1} \ a_{2j})$. If $r = 2n' + 1$, then let $\theta' = (a_1 \ d_1 \ f_1 \ e_1) \prod_{j=1}^{j=n'} (a_{2j} \ a_{2j+1})$. Finally, let $\sigma_z = \Theta \Upsilon \sigma$, where ϵ and ϵ' are the same as the ones defined in Case 1.

2. $r > 1$ and $p = 0$.

Then let $\sigma_z = \Theta \sigma$, where θ and θ' are the same as in (1).

3. $r = 0$ and $p > 1$.

If $p = 2m'$ then if $m' = 1$ let $\epsilon = (e_1 \ b_2 \ b_1 \ c_1 \ c_2)(d_1 \ f_1)$ and if $m' > 1$ then let $\epsilon = (e_1 \ b_2 \ b_1 \ c_1 \ c_2)(d_1 \ f_1) \prod_{j=2}^{j=m'} (b_{2j} \ c_{2j} \ b_{2j-1} \ c_{2j-1})$. On the other hand, if $q = 2m' + 1$, then let $\epsilon' = (e_1 \ f_1 \ d_1 \ b_1 \ c_1) \prod_{j=1}^{j=m'} (b_{2j} \ c_{2j} \ b_{2j+1} \ c_{2j+1})$. Finally, let $\sigma_z = \Upsilon \sigma$.

- Case 5. $r = 0$.

1. $p > 1$ and $q > 1$.

Let $\sigma_z = \Upsilon \Delta \sigma$, where δ , δ' , ϵ and ϵ' are the same as the ones defined in Case 1.

2. $p > 1$ and $q = 0$.

Then let $\sigma_z = \Upsilon \sigma$, where ϵ and ϵ' are the same as the ones defined in Case 1.

3. $p = 0$ and $q > 1$.

Then let $\sigma_z = \Delta \sigma$, where δ and δ' are the same as the ones defined in Case 1.

- Case 6. $p = 0$.

1. $r > 1$ and $q > 1$.

Then let $\sigma_z = \Theta \Delta \sigma$, where δ , δ' , θ and θ' are the same as the ones defined in Case 1.

2. $r > 1$ and $q = 0$.

Then let $\sigma_z = \Theta \sigma$, where θ and θ' are the same as the ones defined in Case 1.

- Case 7. $q = 0$.

We still have only the case where $r > 1$ and $p > 1$. Then let $\sigma_z = \Theta \Upsilon \sigma$, where θ , θ' , ϵ and ϵ' are the same as the ones defined in Case 1.

Note that in each of the above cases, $\text{dist}_T(x_i, \sigma_z(x_i)) \leq 2$ for $i = 1, \dots, m$, $\text{dist}_T(c_l, \sigma_z(c_l)) \leq 4$ for $l = 1, \dots, p$, $\text{dist}_T(f_j, \sigma_z(f_j)) \leq 4$ for $j = 1, \dots, q$ and $\text{dist}_T(\sigma_z(u), \sigma_z(v)) \leq 5$ whenever uv is an edge in G . Thus, it can be easily proved that σ_z is a (T, z) -well 2-placement. ■

Corollary 3.1 *Consider a non star tree containing a vertex x such that $d(x) > 2$. Let $\{x_1, \dots, x_n\}$, $n > 2$, be the neighbors of x . Suppose that $T_{(x_i, x)}$ is a neighbor F -tree of x for $i = 1, \dots, m$, where $2 \leq m < n$. Let T' be the connected component containing x in $T - \{xx_i; i = 1, \dots, m\}$. Suppose that there exists a (T', z) -good 2-placement σ such that $\text{dist}(x, \sigma(x)) \leq 2$, where z is a vertex in T' not necessarily distinct from x , then there exists a (T, z) -good 2-placement say σ_z such that $\sigma_z(v) = \sigma(v)$ for every $v \in V(T')$ and $\text{dist}(x_i, \sigma_z(x_i)) \leq 2$ for $i = 1, \dots, m$.*

Proof. Let G be the graph obtained by the union of all the components that do not contain x in $T - \{xx_i; i = 1, \dots, m\}$. Since σ is a (T', z) -good 2-placement then it is a (T', z) -well 2-placement, and so, by Lemma 3.3, there exists a (T, z) -well 2-placement, say σ_z , such that $\sigma_z(v) = \sigma(v)$ for every $v \in V(T')$, $\text{dist}(x_i, \sigma_z(x_i)) \leq 2$ for $i = 1, \dots, m$ and $\text{dist}(\sigma_z(u), \sigma_z(v))_T \leq 5$ whenever uv is an edge in G . Thus σ_z is a (T, z) -good 2-placement. ■

Lemma 3.4 *Let T be a non star tree and let x be a vertex of T with $N(x) = \{x_1, \dots, x_n\}$, $n \geq 2$. Suppose that there exists a $(T_{(x_i, x)}, x_i)$ -well 2-placement for $i = 1, \dots, p$, where $1 \leq p < n$. Let T' be the connected component containing x in $T - \{xx_i; i = 1, \dots, p\}$. If there exists a (T', z) -well 2-placement, say σ , such that $\text{dist}(x, \sigma(x)) \leq 3$, where z is a vertex of T' , then there exists a (T, z) -well 2-placement, say σ_z , such that $\sigma_z(v) = \sigma(v)$ for every $v \in V(T')$.*

Proof. Let σ_i be a $(T_{(x_i, x)}, x_i)$ -well 2-placement for $i = 1, \dots, p$. Then the 2-placement σ defined as follows:

$$\sigma_z(v) = \begin{cases} \sigma(v) & \text{if } v \in V(T') \\ \sigma_i(v) & \text{if } v \in V(T_{x_i, x}) \text{ for } i = 1, \dots, p \end{cases}$$

is a (T, z) -well 2-placement. ■

Lemma 3.5 *Let T be a non star tree and let x be a vertex of T with $N(x) = \{x_1, \dots, x_n\}$, $n \geq 4$. Let T' be the connected component containing x in $T - \{xx_i; i = 1, \dots, p\}$, $3 \leq p < n$. Suppose that at least two and at most $p - 1$ trees in the set $\{T_{(x_i, x)} : i = 1, \dots, p\}$ are neighbor F -trees of x such that for the remaining non neighbor F -trees in this set there exists a $(T_{(x_i, x)}, x_i)$ -well 2-placement. If there exists a (T', z) -well 2-placement, say σ' , such that $\text{dist}(x, \sigma'(x)) \leq 3$, where z is a vertex of T' not necessarily distinct from x , then there exists a (T, z) -well 2-placement, say σ , such that $\sigma(v) = \sigma'(v)$ for every $v \in V(T')$.*

Proof. Suppose that $T_{(x_i, x)}$ are the neighbor F -trees of x for $i = 1, \dots, k$, where $2 \leq k \leq p - 1$, such that there exists a $(T_{(x_i, x)}, x_i)$ -well 2-placement for $i = k + 1, \dots, p$. Let T'' be the connected component containing x in $T - \{xx_i; i = k + 1, \dots, p\}$. Since $\text{dist}(x, \sigma'(x)) \leq 3$ and since $T_{(x_i, x)}$ are neighbor F -trees of x for $i = 1, \dots, k$, then, by Lemma 3.3, there exists a (T'', z) -well 2-placement, say σ'' , such that $\sigma'(v) = \sigma''(v)$ for every $v \in V(T')$. Again, since $\text{dist}(x, \sigma''(x)) \leq 3$ and there exists a $(T_{(x_i, x)}, x_i)$ -well 2-placement for $i = k + 1, \dots, p$, then, by the previous Lemma, there exists a (T, z) -well 2-placement, say σ , such that $\sigma(v) = \sigma''(v)$ for every $v \in V(T'')$. ■

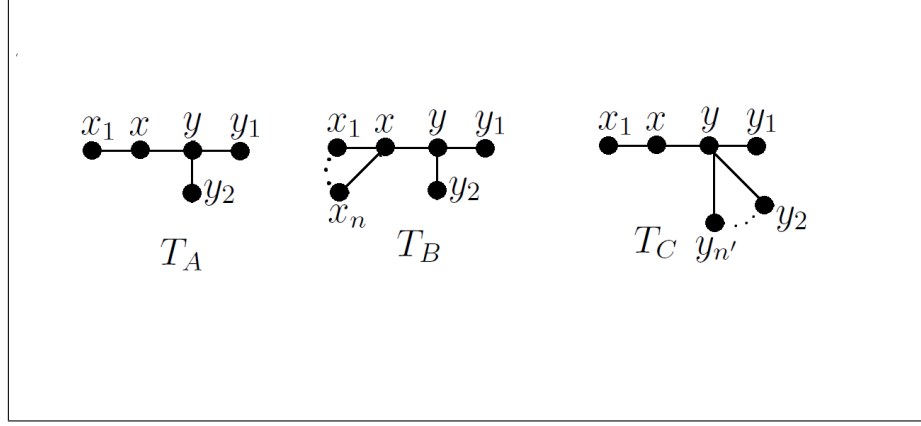


Figure 1

Lemma 3.6 *Let T be one of the trees in Fig. 1 such that $n \geq 2$ and $n' \geq 3$. Then there exists a (T, x_1) -good 2-placement and a (T, v) -well 2-placement for every vertex v of T .*

Proof. For each tree T in Fig. 1, we give below a (T, x_1) -good 2-placement and a (T, v) -well 2-placement for every vertex v of T .

$\sigma = (x_1 \ x \ y_1 \ y_2 \ y)$ is a (T_A, x_1) -good 2-placement and a (T_A, v) -well 2-placement for every $v \in V(T_A)$.

There are four choices for choosing a vertex v of T_B , either x_1 , x , y or y_1 . If $n = 2$, then $\sigma = (x_2 \ y_2)(y_1 \ y \ x_1 \ x)$ is a (T_B, x_1) -good 2-placement and a (T_B, v) -well 2-placement, for every $v \in \{x_1, x, y, y_1\}$. Otherwise, suppose that $n \geq 3$. If $n = 2k + 1$ for some $k \in \mathbb{N}$, then $\sigma = (x_1 \ x \ y_1 \ y_2 \ y) \prod_{i=1}^{i=k} (x_{2i} \ x_{2i+1})$ is a (T_B, x_1) -good 2-placement and a (T_B, v) -well 2-placement $\forall v \in V(T_B)$. On the other hand, if $n = 2k$ for some $k \in \mathbb{N}$, then $\sigma = (x_2 \ y_2)(y_1 \ y \ x_1 \ x) \prod_{i=2}^{i=k} (x_{2i-1} \ x_{2i})$ is a (T_B, x_1) -good 2-placement and a (T_B, v) -well 2-placement, for every $v \in \{x_1, x, y, y_1\}$.

Finally, if $n' = 2k + 1$ for some $k \in \mathbb{N}$, then $\sigma = (x_1 \ x \ y_1 \ y) \prod_{i=1}^{i=k} (y_{2i} \ y_{2i+1})$ is a (T_C, x_1) -good 2-placement and a (T_C, v) -well 2-placement $\forall v \in V(T_C)$.

And if $n' = 2k$ for some $k \in \mathbb{N}$, $k \geq 2$, then $\sigma = (x_1 \ x \ y_1 \ y_2 \ y) \prod_{i=2}^{i=k} (y_{2i-1} \ y_{2i})$ is a (T_C, x_1) -good 2-placement and a (T_C, v) -well 2-placement for every $v \in V(T_C)$. ■

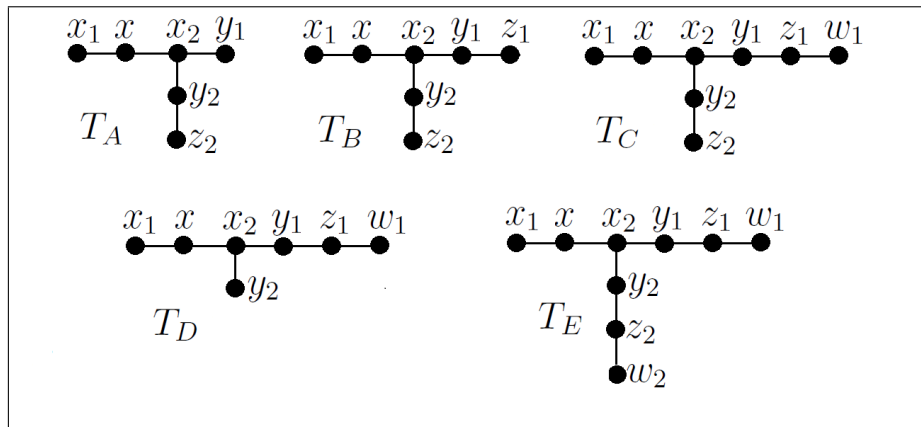


Figure 2

Lemma 3.7 *Let T be one of the trees in Fig. 2. Then there exists a (T, x) -well 2-placement and a (T, x_1) -well 2-placement.*

Proof. For each pair (T, x) and (T, x_1) in Fig. 2, we give below a (T, x) -well 2-placement σ_x and a (T, x_1) -well 2-placement σ_{x_1} .

For T_A , $\sigma_x = \sigma_{x_1} = (x_1 x y_2)(x_2 y_1 z_2)$.

For T_B , $\sigma_x = \sigma_{x_1} = (x_1 x y_2)(y_1 z_2)(x_2 z_1)$.

For T_C , $\sigma_x = \sigma_{x_1} = (x_1 x y_2)(y_1 z_2)(x_2 w_1 z_1)$.

For T_D , $\sigma_x = \sigma_{x_1} = (x_1 x y_2)(x_2 y_1 w_1 z_1)$.

For T_E , $\sigma_x = \sigma_{x_1} = (x_1 x y_2)(z_2 w_2 y_1)(z_1 x_2 w_1)$. ■

Lemma 3.8 *Consider a non star tree T containing an edge xy such that $d_T(x) = 1$ and $d_T(y) \geq 3$. Suppose that all the neighbor trees of y distinct from $T_{(x,y)}$ are neighbor F -trees not isomorphic to P_1 , then there exists a (T, x) -well 2-placement.*

Proof. Let $\{y_1, \dots, y_n\}$, where $n \geq 2$, be the neighbors of y distinct from x . We need to consider the following two cases concerning the degree of y :

1. $d(y) > 3$.

Let T_0 be the connected component containing x in $T - \{yy_i; i = 2, \dots, n\}$, then T_0 is isomorphic to P_4 or P_5 , and so, by lemma 3.1, there exists a (T_0, x) -well path 2-placement, say σ_0 , such that $\text{dist}(v, \sigma_0(v)) \leq 3$ for every vertex v of T_0 . Since $T_{(y_i, y)}$ are neighbor F -trees of y for $i = 2, \dots, n$, then, by Lemma 3.3, there exists a (T, x) -well 2-placement.

2. $d(y) = 3$

Then T is isomorphic to one of the following trees in Fig. 3. For each pair (T, x) in Fig.

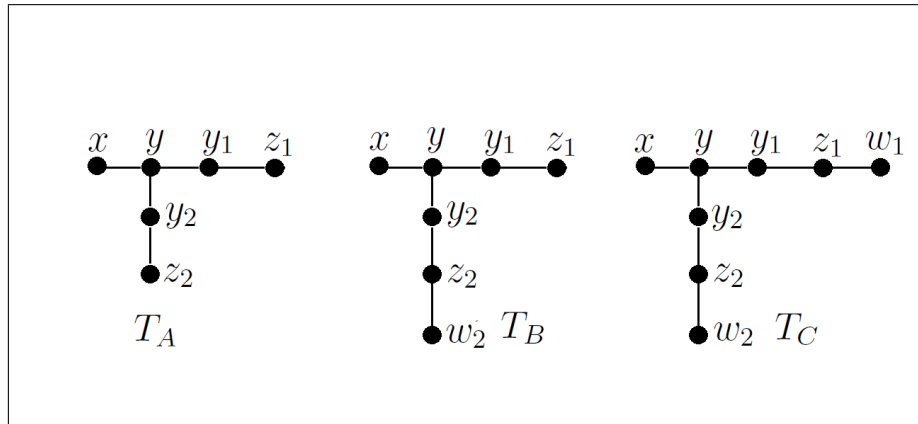


Figure 3

3, we give below a (T, x) -well 2-placement σ :

$\sigma = (x y z_2)(y_1 z_1 y_2)$ is a (T_A, x) -well 2-placement.

$\sigma = (x y z_1)(y_1 z_2)(y_2 w_2)$ is a (T_B, x) -well 2-placement.

$\sigma = (x y z_1)(y_1 z_2 w_1)(w_2 y_2)$ is a (T_C, x) -well 2-placement.

■

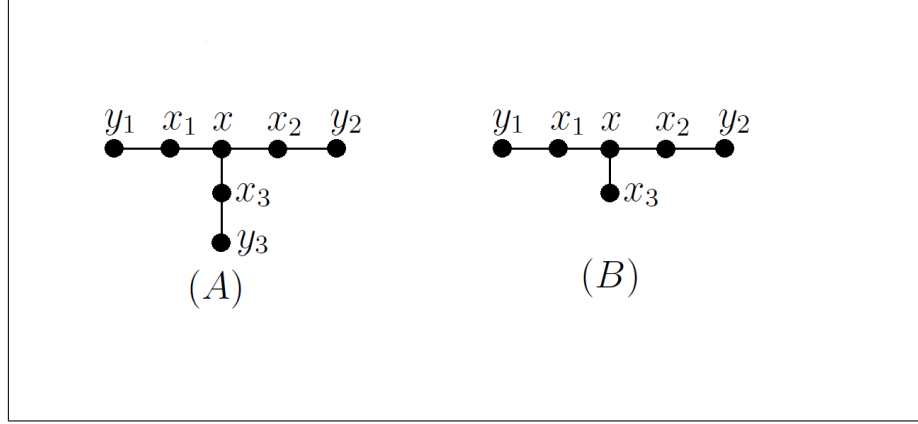


Figure 4

Lemma 3.9 Consider a non star tree containing a vertex x with $d(x) \geq 3$. Suppose that all the neighbor trees of x are isomorphic either to P_1 or P_2 such that x has at most one neighbor tree isomorphic to P_1 , then there exists a (T, x) -well 2-placement.

Proof. Let $N(x) = \{x_1, \dots, x_n\}$, $n \geq 3$. If $d(x) > 3$, then let T_0 be the connected component containing x in $T - \{xx_i; i = 3, \dots, n\}$, then T_0 is isomorphic either to P_4 or P_5 . Hence, by Lemma 3.1, there exists a (T_0, x) -well path 2-placement, say σ_0 . Since $\text{dist}(x, \sigma_0(x)) \leq 2$ and $T_{(x_i, x)}$ are neighbor F -trees for $i = 3, \dots, n$, then, by Lemma 3.3, there exists a (T, x) -well 2-placement. On the other hand, if $d(x) = 3$, then T is isomorphic to one of the trees in Fig. 4. For each pair (T, x) in Fig. 4, we give below a (T, x) -well 2-placement σ :

For T_A , $\sigma = (x y_3)(y_2 x_2 x_1)(x_3 y_1)$

For T_B , $\sigma = (x y_1)(x_1 y_2)(x_3 x_2)$

■

Lemma 3.10 Consider a non star tree T containing an edge xy such that $d_T(x) = 1$ and $d_T(y) \geq 3$. Suppose that all the neighbor trees of y other than $T_{(x, y)}$ are neighbor F -trees not isomorphic to P_1 . Then there exists a (T, x) -good 2-placement.

Proof. Let $\{y_1, \dots, y_n\}$ be the set of neighbors of y distinct from x , where $n \geq 2$. We are going to study two cases:

- y has a neighbor tree isomorphic to P_2 .

Without loss of generality, suppose that $T_{(y_1, y)} = y_1 z_1$. If $n > 2$, then let $T_0 = T - \{yy_i : i = 2, \dots, n\}$ $\sigma_0 = (x y z_1 y_1)$ be a (T_0, x) -good 2-placement. Since $\text{dist}(y, \sigma_0(y)) = 2$, then, by Corollary 3.1, there exists a (T, x) -good 2-placement. Otherwise, that is, $n = 2$, then T is either isomorphic to T_A or T_B in Fig. 3. For each case, we give below a (T, x) -good 2-placement σ :

For T_A , $\sigma = (x y z_1)(y_1 y_2 z_2)$.

For T_B , $\sigma = (x y z_1 y_1 z_2)(y_2 w_2)$.

- y has a neighbor tree isomorphic to P_3 .

Without loss of generality, suppose that $T_{(y_1, y)} = y_1 z_1 w_1$. If $n > 2$, then let $T_0 = T - \{yy_i : i = 2, \dots, n\}$ and $\sigma_0 = (x y z_1 w_1 y_1)$ be a (T_0, x) -good 2-placement. Since $\text{dist}(y, \sigma_0(y)) = 2$, then, by Corollary 3.1, there exists a (T, x) -good 2-placement.

Otherwise, that is, $n = 2$, then T is isomorphic either to T_B or T_C in Fig. 3. We showed above that there exists a (T_B, x) -good 2-placement. For T_C , $\sigma = (x \ y \ z_1 \ w_1 \ y_1 \ y_2 \ w_2 \ z_2)$ is a (T_C, x) -good 2-placement.

■

Lemma 3.11 *Let T be one of the trees in Fig. 5 such that $n \geq 2$. Then there exists a (T, x) -good 2-placement σ_x such that $\text{dist}(a_1, \sigma_x(a_1)) \leq 2$ whenever T is isomorphic to T_E , T_F or T_G .*

Proof. For each pair (T, x) in Fig. 5, we give below a (T, x) -good 2-placement σ_x .

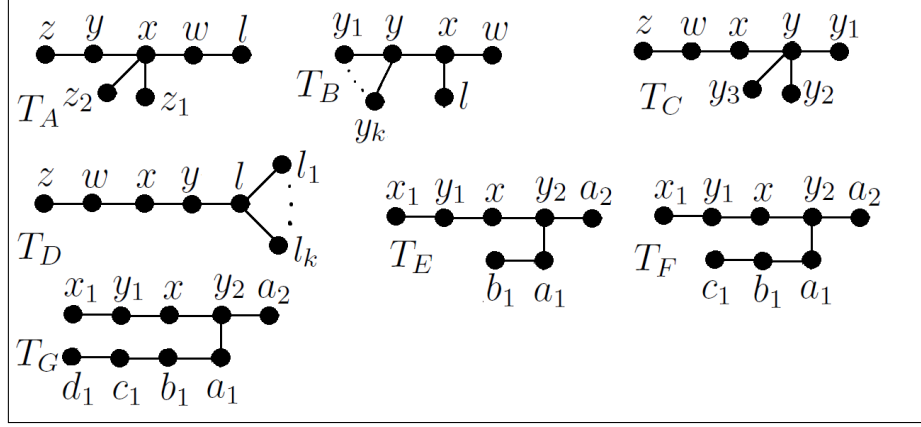


Figure 5

For T_A , $\sigma_x = (x \ z_1 \ z_2 \ w \ l \ y \ z)$

For T_B , if $k = 2p$ for some $p \in \mathbb{N}$, then $\sigma_x = (x \ w \ l \ y \ y_1 \ y_2)$ if $k = 2$, and if $k > 2$, then $\sigma_x = (x \ w \ l \ y \ y_1 \ y_2) \prod_{i=2}^{i=p} (y_{2i-1} \ y_{2i})$. If $k = 2p + 1$, then $\sigma_x = (x \ w \ l \ y \ y_1) \prod_{i=1}^{i=p} (y_{2i+1} \ y_{2i})$

For T_C , $\sigma_x = (x \ y \ w \ z)(y_1 \ y_2 \ y_3)$.

For T_D , if $k = 2p$ for some $p \in \mathbb{N}$, then $\sigma_x = (x \ y \ l_1 \ l_2 \ l \ w \ z)$ if $k = 2$, and if $k > 2$, $\sigma_x = (x \ y \ l_1 \ l_2 \ l \ w \ z) \prod_{i=2}^{i=p} (l_{2i-1} \ l_{2i})$. If $n = 2p + 1$, then $\sigma_x = (x \ y \ l_1 \ l \ w \ z) \prod_{i=1}^{i=p} (l_{2i+1} \ l_{2i})$.

For T_E , $\sigma_x = (x \ y_2 \ y_1 \ x_1)(a_1 \ b_1 \ a_2)$.

For T_F , $\sigma_x = (x \ y_2 \ y_1 \ x_1)(a_1 \ c_1 \ b_1 \ a_2)$.

For T_G , $\sigma_x = (x \ y_2 \ y_1 \ x_1)(a_1 \ c_1 \ d_1)(b_1 \ a_2)$.

■

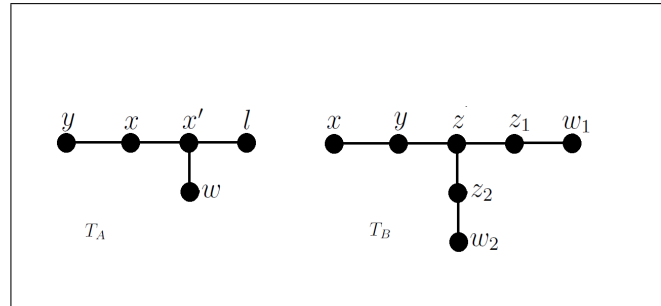


Figure 6

Lemma 3.12 *For the trees shown in Fig. 6, there exists a (T_A, x) -good 2-placement, a (T_A, x') -good 2-placement and a (T_B, x) -good 2-placement.*

Proof. $\sigma = (x \ y \ x' \ w \ l)$ is a (T_A, x) -good 2-placement and a (T_A, x') -good 2-placement. $\sigma = (x \ y \ z_1 \ w_1 \ z_2)(w_2 \ z)$ is a (T_B, x) -good 2-placement. ■

Proof of Theorem 1.7.

The proof is by induction on the order of T . For $n = 4$, there is only one non star tree, which is P_4 , then, by Lemma 3.1, there exists a (T, v) -well 2-placement for every $v \in V(P_4)$. Suppose that the theorem holds for $n' < n$, $n \geq 5$, and let T be a non star tree of order n . Let x be a vertex of T . If T is a path, then, by Theorem 3.1, there exists a (T, x) -well path 2-placement. Otherwise, let v be a vertex of T and let $\{v_1, \dots, v_m\}$, $m \geq 2$, be the neighbors of v . Suppose that $\{v_1, \dots, v_p\}$, $1 \leq p < m$, are enumerated in such a way that x and v are in the same connected component in $T - \{vv_i : i = 1, \dots, p\}$ and let T' be this connected component. Note that x and v may be the same vertex.

Claim 1. If there exists a (T', x) -well 2-placement σ such that $\text{dist}(v, \sigma(v)) \leq 3$ and $T_{(v_i, v)}$ are non star trees for $i = 1, \dots, p$, then there exists a (T, x) -well 2-placement.

Proof. Since $T_{(v_i, v)}$ is a non star tree for $i = 1, \dots, p$, then, by induction, there exists a $(T_{(v_i, v)}, v_i)$ -well 2-placement. Thus, by Lemma 3.4, there exists a (T, x) -well 2-placement.

Claim 2. If $2 \leq p < m$, at least two neighbor trees in the set $\{T_{(v_i, v)} : i = 1, \dots, p\}$ are neighbor F -trees of v such that the remaining neighbor trees in the set are non star trees and if there exists a (T', x) -well 2-placement σ such that $\text{dist}(v, \sigma(v)) \leq 3$, then there exists a (T, x) -well 2-placement.

Proof. If $T_{(v_i, v)}$ is a non star tree for some i , $1 \leq i \leq p$, then, by induction, there exists a $(T_{(v_i, v)}, v_i)$ -well 2-placement. Thus, since at least two neighbor trees in the set $\{T_{(v_i, v)} : i = 1, \dots, p\}$, are neighbor F -trees of v , there exists, by Lemma 3.3 and Lemma 3.5, a (T, x) -well 2-placement.

From now on, we shall assume that we can't apply neither Claim 1 nor Claim 2 on any vertex v of T . We will study two cases concerning the degree of x .

Case 1. $d(x) = 1$.

Let y be the father of x . If y is the father of another leaf, say α , then let $T' = T - \{x\}$. Note that T' is a non star tree since otherwise T is a star. So, there exists a (T', α) -well 2-placement, say σ' . The 2-placement σ defined as follows:

$$\sigma(v) = \begin{cases} \sigma'(v) & \text{if } v \in V(T) - \{\alpha, x\} \\ x & \text{if } v = \alpha \\ \sigma'(\alpha) & \text{if } v = x \end{cases}$$

is a (T, x) -well 2-placement. Otherwise, suppose that y is the father of the leaf x only. If there exists a set of leaves, say $\{\alpha_1, \dots, \alpha_k\}$, $k \geq 2$, such that all of these leaves have the same father, say β , with $d(\beta) = k + 1$, then let $T' = T - \{\alpha_1, \dots, \alpha_k\}$. If T' is a non star tree then

there exists a (T', x) -well 2-placement, say σ_x , such that $\text{dist}(\beta, \sigma_x(\beta)) \leq 4$, since β is a leaf in T' , and so $\sigma = \sigma_x(\alpha_1 \dots \alpha_k)$ is a (T, x) -well 2-placement. Else, that is T' is a star, then T is isomorphic to T_A in Fig. 1 if $k = 2$, and if $k > 2$ then T is isomorphic to T_C in Fig. 1 for $n' = k$. Thus, by Lemma 3.6, there exists a (T, x) -well 2-placement. Otherwise, suppose that the set $\{\alpha_1, \dots, \alpha_k\}$ doesn't exist. Hence, we can remark that for any edge ab in $E(T)$, $T_{(a,b)}$ is either a neighbor F -tree of b or a non star tree. Since T is not a path, then there exists a vertex in T with degree strictly greater than two. Let z be the first vertex away from x such that $d(z) > 2$, z' be the nearest neighbor of z to x and let $\{z_1, \dots, z_m\}$, $m \geq 2$, be the neighbors of z distinct from z' . Note that $T_{(z',z)}$ is a path and $T_{(z,z')}$ is a non star tree since otherwise the set $\{\alpha_1, \dots, \alpha_k\}$, which is described above, exists. $T_{(z',z)}$ is a path of length at most two, since otherwise there exists, by Theorem 3.1, a $(T_{(z',z)}, x)$ -well path 2-placement, say σ_x , such that $\text{dist}(z', \sigma_x(z')) \leq 3$, and so we can apply Claim 1 on z' . Suppose first that the path $T_{(z',z)}$ is of length zero, that is x is a neighbor of z , then z is the father of the leaf x only. If z has a non neighbor F -tree, then suppose that $T_{(z_1,z)}$ is a non star tree. Since $T_{(z,z_1)}$ is a non star tree, then there exists a $(T_{(z,z_1)}, x)$ -well 2-placement σ' such that $\text{dist}(z, \sigma'(z)) \leq 3$, and so we can apply Claim 1 on z , a contradiction. Hence, all the neighbor trees of z are neighbor F -trees and so there exists a (T, x) -well 2-placement by Lemma 3.8. Suppose now that the path $T_{(z',z)}$ is of length two or one. If z has a unique neighbor F -tree or at least three neighbor F -trees other than $T_{(z',z)}$, then suppose that $T_{(z_1,z)}$ is a neighbor F -tree and let T' be the connected component containing x in $T - \{zz_i; i = 2, \dots, m\}$. Thus, T' is a path of length at least three and at most six and so, by Lemma 3.1, there exists a (T', x) -well path 2-placement σ' such that $\text{dist}(v, \sigma'(v)) \leq 3$ for every vertex v of T' . Hence, we can apply Claim 1 or Claim 2 on z , a contradiction. Thus, z has no neighbor F -tree or z has only two neighbor F -trees distinct from $T_{(z',z)}$. If $T_{(z',z)}$ is a path of length two, then let T' be the connected component containing x in $T - \{zz_i; i = 1, \dots, m\}$. Since T' is a path of length three, then there exists a (T', x) -well path 2-placement σ' , by Lemma 3.1, such that $\text{dist}(z, \sigma'(z)) \leq 3$, and so we can apply Claim 1 or Claim 2 on z , a contradiction. Thus, $T_{(z',z)}$ is a path of length one, that is z' is the father of x . If z has no neighbor F -tree distinct from $T_{(z',z)}$, then each neighbor of z has at least two neighbors. Let $\{a_1, \dots, a_q\}$, $q \geq 1$, be the neighbors of z_1 distinct from z . If z_1 has at least two neighbor F -trees or no neighbor F -tree distinct from $T_{(z,z_1)}$, then let T' be the connected component containing x in $T - \{zz_j, z_1a_i; j = 2, \dots, m \text{ and } i = 1, \dots, q\}$. Thus, T' is a path of length three and so, by Lemma 3.1, there exists a (T', x) -well path 2-placement σ' such that $\text{dist}(v, \sigma'(v)) \leq 3$ for every vertex v of T' . Since $T_{(z_j,z)}$ are non neighbor F -trees for $j = 2, \dots, m$, then, by Lemma 3.4, there exists a (T'', x) -well 2-placement σ'' , where T'' is the connected component containing x in $T - \{z_1a_i; i = 1, \dots, q\}$, such that $\sigma''(v) = \sigma'(v)$ for every vertex v of T'' and so we can either apply Claim 1 or Claim 2 on z_1 , a contradiction. Thus, z_1 has a unique neighbor F -tree distinct from $T_{(z,z_1)}$, say $T_{(a_1,z_1)}$. Let T' be the connected component containing x in $T - \{zz_j, z_1a_i; j = 2, \dots, m \text{ and } i = 2, \dots, q\}$, then T' is a path of length at least four and at most six, and so, by Lemma 3.1, there exists a (T', x) -well path 2-placement σ' such that $\text{dist}(v, \sigma'(v)) \leq 3$ for every vertex v of T' . Thus, by Lemma 3.4, there exists a (T'', x) -well 2-placement σ'' , where T'' is the connected component containing x in $T - \{z_1a_i; i = 2, \dots, q\}$, such that $\sigma''(v) = \sigma'(v)$ for every v of T' and so we can either apply Claim 1 on z_1 , a contradiction. Thus, z has only two neighbor F -trees distinct from $T_{(z',z)}$. If $m > 2$, then suppose that $T_{(z_1,z)}$ and $T_{(z_2,z)}$ are the neighbor F -trees of z . The tree T' , which is the connected component containing x in $T - \{zz_i; i = 3, \dots, m\}$, is either isomorphic to one of the trees in Fig. 2 or to the tree T_A in Fig. 1, and so, by Lemma 3.6 and Lemma 3.7, there

exists a (T', x) -well 2-placement σ' such that $\text{dist}(z, \sigma'(z)) \leq 3$. Thus, we can apply Claim 1 on z , a contradiction. Hence, $m = 2$ and so T is isomorphic to one of the trees in Fig. 2, and so, by Lemma 3.7, there exists a (T, x) -well 2-placement.

Case 2. $d(x) > 1$.

If x or any of its neighbors is a father of at least two leaves, say α_1 and α_2 , then let $T' = T - \{\alpha_1, \alpha_2\}$. T' is a star, since otherwise there exists a (T', x) -well 2-placement, say σ' , such that $\text{dist}(v, \sigma'(v)) \leq 3$ for every $v \in \{x \cup N(x)\}$ and so we can apply Claim 2 on the father of α_1 and α_2 , a contradiction. Hence, T is isomorphic to T_A or T_B in Fig. 1, and so, by Lemma 3.6, there exists a (T, x) -well 2-placement. Else, suppose that neither x nor any of its neighbors is a father of at least two leaves. If there exists a set of leaves, say $\{\alpha_1, \dots, \alpha_m\}$, $m \geq 2$, such that all of the leaves have the same father, say β , with $d(\beta) = m + 1$, then let $T' = T - \{\alpha_1, \dots, \alpha_m\}$. Note that T' is a non star tree since neither x nor any of its neighbors is a father of at least two leaves. Hence, there exists a (T', x) -well 2-placement, say σ_x , such that $\text{dist}(\beta, \sigma_x(\beta)) \leq 4$ since β is an end vertex in T' . Then $\sigma = \sigma_x(\alpha_1 \dots \alpha_m)$ is a (T, x) -well 2-placement. Otherwise, suppose that the set of leaves $\{\alpha_1, \dots, \alpha_m\}$ doesn't exist. Since T is not a path and all the previous cases are not satisfied then there exists $y \in N(x)$ such that $T_{(x,y)}$ is a non star tree. There exists no neighbor y of x such that $T_{(x,y)}$ and $T_{(y,x)}$ are non star trees, since otherwise there exists a $(T_{(x,y)}, x)$ -well 2-placement, and so we can apply Claim 1 on x . If there exists $y \in N(x)$ such that $T_{(x,y)}$ is a non star tree and $T_{(y,x)}$ is a path of length two, then all the neighbor trees of x are neighbor F -trees. And since T is not a path then $d(x) > 2$. Let $\{y_1, \dots, y_m\}$, $m \geq 2$, be the neighbors of x distinct from y , T_0 be the connected component containing x in $T - \{xy_i; i = 1, \dots, m\}$. Then T_0 is a path of length three and so there exists, by Lemma 3.1, a (T_0, x) -well 2-placement. Hence, we can apply Claim 2 on x , a contradiction. Thus, there exists no neighbor y of x such that $T_{(x,y)}$ is a non star tree and $T_{(y,x)}$ is a path of length two. If there exists $y \in N(x)$ such that $T_{(x,y)}$ is a non star tree and $T_{(y,x)}$ is a path of length one, then if $d(x) = 2$, let y_1 be the neighbor of x distinct from y . $T_{(y_1,x)}$ is not a neighbor F -tree of x since T is not a path. Let $\{b_1, \dots, b_r\}$, $r \geq 1$, be the neighbors of y_1 distinct from x and let T_0 be the connected component containing x in $T - \{y_1 b_i; i = 1, \dots, r\}$. Then T_0 is a path of length three, and so there exists, by Lemma 3.1, a (T_0, x) -well path 2-placement σ_0 such that $\text{dis}(y, \sigma_0(y)) \leq 3$. Thus, y_1 has a unique neighbor F -tree distinct from $T_{(x,y_1)}$, since otherwise we can apply either Claim 1 or Claim 2 on y_1 . Suppose that $T_{(b_1,y_1)}$ is that tree and let T' be the connected component containing x in $T - \{y_1 b_i; i = 2, \dots, r\}$. Hence, T' is a path of length at least four and at most six, and so, by Lemma 3.1, there exists a (T', x) -well path 2-placement, say σ' , such that $\text{dist}(y, \sigma'(y)) \leq 3$. Thus, we can apply Claim 1 on y_1 , a contradiction. Hence, $d(x) > 2$, and so all the neighbor trees of x are paths of length one with at most one is a path of length zero. Hence, by Lemma 3.9, there exists a (T, x) -well 2-placement. Otherwise, suppose that there exists no $y \in N(x)$ such that $T_{(x,y)}$ is a non star tree and $T_{(y,x)}$ is a path of length one. Then, there exists $y \in N(x)$ such that $T_{(x,y)}$ is a non star tree, $T_{(y,x)}$ is a path of length zero and $d(x) = 2$. Let $N(x) = \{y, y'\}$. If $d(y') = 2$, then let a be the neighbor of y' distinct from x and let $\{a_1, \dots, a_k\}$, $k \geq 1$, be the neighbors of a distinct from y' . Let T_0 be the connected component containing x in $T - \{aa_i; i = 1, \dots, k\}$, then T_0 is a path of length three, and so, by Lemma 3.1, there exists a (T_0, x) -well path 2-placement σ_0 such that $\text{dist}(a, \sigma_0(a)) \leq 3$. Thus, a has a unique neighbor F -tree distinct from $T_{(y',a)}$, since otherwise we can apply either Claim 1 or Claim 2 on a . Suppose that $T_{(a_1,a)}$ is that tree and let T' be the connected component containing x in $T - \{aa_i; i = 2, \dots, k\}$. Then T' is a path of length at

least four and at most six, and so, by Lemma 3.1, there exists a (T', x) -well path 2-placement, say σ' , such that $\text{dist}(a, \sigma'(a)) \leq 3$. Thus, we can apply Claim 1 on a , a contradiction. Hence, $d(y') > 2$. let $\{y'_1, \dots, y'_l\}$, $l \geq 2$, be the neighbors of y' distinct from x . If y' has a non neighbor F -tree then suppose that $T_{(y'_1, y')}$ is that tree. Note that $T_{(y', y'_1)}$ is a non star tree and so there exists a $(T_{(y', y'_1)}, x)$ -well 2-placement say σ' such that $\text{dist}(y', \sigma'(y')) \leq 3$. Thus, Claim 1 is applied on y' , a contradiction. Hence, all the neighbor trees of y' are neighbor F -trees. If $d(y') > 3$, then let T' be the connected component containing x in $T - \{y'_i y'_i : i = 2, \dots, l\}$. Since T' is a non star tree, then there exists a (T', x) -well 2-placement and so we can apply Claim 2 on y' , a contradiction. Thus, $d(y') = 3$ and T is isomorphic to one of the trees in Fig. 2, and so, by Lemma 3.7, there exists a (T, x) -well 2-placement. \square

Proof of Corollary 1.3.

Let $T_0 = T - \{\alpha_1, \dots, \alpha_{m_T}\}$, where $\{\alpha_1, \dots, \alpha_{m_T}\}$ is the maximal set of leaves that can be removed from T in such a way that the obtained tree is a non star one. Since T_0 is a non star tree then there exists a (T_0, x) -well 2-placement for some x in T_0 . We define a packing of T into T^6 , say σ , as follows:

$$\sigma(v) = \begin{cases} \sigma' & \text{if } v \in V(T') \\ \alpha_i & \text{if } v = \alpha_i \text{ for } i = 1, \dots, m_T \end{cases}$$

Label α_i by i , for $i = 1, \dots, m_T$. Let r be the number of cycles of σ and let $\sigma_1, \dots, \sigma_r$ be those cycles. Remark that $r \geq \lceil \frac{n-m_t}{5} \rceil$. Label the vertices of each cycle σ_i by $m_T + i$ for $i = 1, \dots, r$. Hence, we obtain an $(m_T + r)$ -labeled packing of T into T^6 and so $w^6(T) \geq m_T + \lceil \frac{n-m_t}{5} \rceil$. \square

Proof of Theorem 1.8.

The proof is by induction on the order of T . For $n = 4$ there is only one non star tree, which is P_4 . By lemma 3.2, there exists a (T, v) -good 2-placement for every $v \in V(P_4)$. Suppose that the theorem holds for $n' < n$, $n \geq 5$, and let T be a non star tree of order n . Let x be a vertex of T such that x is not a bad vertex. If T is a path, then, by Theorem 3.2, there exists a (T, x) -good path 2-placement. Else, we will study two cases concerning the degree of x :

Case 1. $d(x) = 1$.

Let y be the father of x . If y is a father of another leaf α , then let $T' = T - \{x\}$. Note that T' is a non star tree since otherwise T is a star. Hence, there exists a (T', α) -good 2-placement, say σ' . The 2-placement σ defined as follows:

$$\sigma(v) = \begin{cases} \sigma'(v) & \text{if } v \in V(T') - \{\alpha\} \\ \sigma'(\alpha) & \text{if } v = x \\ x & \text{if } v = \alpha \end{cases}$$

is a (T, x) -good 2-placement. Otherwise, suppose that y is the father of the leaf x only. If there exists a set of leaves, say $\{\alpha_1, \dots, \alpha_m\}$, $m \geq 2$, such that all of these leaves have the same father, say β , with $d(\beta) = m + 1$, then let $T' = T - \{\alpha_1, \dots, \alpha_m\}$. If T' is a non star tree, then there exists a (T', x) -good 2-placement, say σ_x , such that $\text{dist}(\beta, \sigma_x(\beta)) \leq 4$ since β is a leaf in T' . Hence, $\sigma = \sigma_x(\alpha_1 \dots \alpha_m)$ is a (T, x) -good 2-placement. And if T' is a star then T is isomorphic to T_A or T_C in Fig. 1, and so, by Lemma 3.6, there exists a (T, x) -good 2-placement.

Otherwise, suppose that the set $\{\alpha_1, \dots, \alpha_m\}$ doesn't exist. Hence, we can remark that for any edge ab in $E(T)$, $T_{(a,b)}$ can be either a neighbor F -tree of b or a non star tree. Let z be the first vertex away from x such that $d(z) > 2$. z exists since T is not a path. Let z' be the nearest neighbor of z to x and let $\{z_1, \dots, z_m\}$, $m \geq 2$, be the neighbors of z distinct from z' . Note that $T_{(z',z)}$ is a path and $T_{(z,z')}$ is a non star tree, since otherwise the set of leaves described above exists. If $T_{(z',z)}$ is a path of length at least three, then let σ_x be a $(T_{(z',z)}, x)$ -good path 2-placement and let σ_z be a $(T_{(z,z')}, z)$ -good 2-placement if z is not a bad vertex in $T_{(z,z')}$, and if it is, then let σ_z be a $(T_{(z,z')}, z'')$ -good 2-placement, where z'' is a neighbor of z in $T_{(z,z')}$. Since $\text{dist}(z', \sigma_x(z')) \leq 2$, then the 2-placement σ defined as follows:

$$\sigma(v) = \begin{cases} \sigma_x(v) & \text{if } v \in V(T_{(z',z)}) \\ \sigma_z(v) & \text{if } v \in V(T_{(z,z')}) \end{cases}$$

is a (T, x) -good 2-placement. Otherwise, $T_{(z',z)}$ is a path of length at most two. If $T_{(z',z)}$ is a path of length two, then let σ' be a $(T_{(z',y)}, z')$ -good 2-placement. The 2-placement σ defined as follows:

$$\sigma(v) = \begin{cases} \sigma'(v) & \text{if } v \in V(T) - \{x, y, z'\} \\ x & \text{if } v = z' \\ y & \text{if } v = x \\ \sigma'(z') & \text{if } v = y \end{cases}$$

is a (T, x) -good 2-placement. Else, if $T_{(z',z)}$ is a path of length one, that is, x is a neighbor of z' , then if z is not a bad vertex in $T_{(z,z')}$, let σ' be a $(T_{(z,z')}, z)$ -good 2-placement. The 2-placement σ defined as follows:

$$\sigma(v) = \begin{cases} \sigma'(v) & \text{if } v \in V(T) - \{x, z, z'\} \\ x & \text{if } v = z \\ z' & \text{if } v = x \\ \sigma'(z) & \text{if } v = z' \end{cases}$$

is a (T, x) -good 2-placement. And if z is a bad vertex in $T_{(z,z')}$ then T is isomorphic to T_B in Fig. 6, and so, by Lemma 3.12, there exists a (T, x) -good 2-placement. Otherwise, $T_{(z',z)}$ is a path of length zero, that is, x is a neighbor of z . If there exists a neighbor of z , say z_1 , such that $T_{(z_1,z)}$ is not a neighbor F -tree of z , then let σ_{z_1} be a $(T_{(z_1,z)}, z_1)$ -good 2-placement if z_1 is not a bad vertex in $T_{(z_1,z)}$, and if it is, then let σ_{z_1} be a $(T_{(z_1,z)}, w_1)$ -good 2-placement, where w_1 is a neighbor of z_1 in $T_{(z_1,z)}$. $T_{(z,z_1)}$ is a non star tree since z is the father of the leaf x only, then there exists a $(T_{(z,z_1)}, x)$ -good 2-placement, say σ_x , such that $\text{dist}(z, \sigma_x(z)) \leq 2$. We define a (T, x) -good 2-placement σ as follows:

$$\sigma(v) = \begin{cases} \sigma_{z_1}(v) & \text{if } v \in V(T_{(z_1,z)}) \\ \sigma_x(v) & \text{if } v \in V(T_{(z,z_1)}) \end{cases}$$

Otherwise, suppose that all the neighbor trees of z are neighbor F -trees. Then, by Lemma 3.10, there exists a (T, x) -good 2-placement.

Case 2. $d(x) > 1$.

If x or any of its neighbors is a father of at least two leaves distinct from x , say α_1 and α_2 , then let $T' = T - \{\alpha_1, \alpha_2\}$. If T' is a non star tree such that x is not a bad vertex in T' then let σ_x be a (T', x) -good 2-placement. Then $\sigma = \sigma_x(\alpha_1 \alpha_2)$ is a (T, x) -good 2-placement. If x is a bad vertex in T' then T is isomorphic to T_A or T_C in Fig. 5, and so, by Lemma 3.11, there exists a (T, x) -good 2-placement. And if T' is a star then T is isomorphic either to T_B in Fig. 5 or to T_A in Fig. 6, and so, by Lemma 3.11 and Lemma 3.12, there exists a (T, x) -good 2-placement. Otherwise, x and each of its neighbors is the father of at most one leaf. If there exists a set of leaves, say $\{\alpha_1, \dots, \alpha_m\}$, $m \geq 2$, such that all of the leaves have the same father, say β , with $d(\beta) = m + 1$, then let $T' = T - \{\alpha_1, \dots, \alpha_m\}$. Note that T' is a non star tree since neither x nor any of its neighbors is a father of at least two leaves. If x is not a bad vertex in T' , then there exists a (T', x) -good 2-placement, say σ_x , such that $\text{dist}(\beta, \sigma_x(\beta)) \leq 4$ since β is a leaf in T' . Thus $\sigma = \sigma_x(\alpha_1 \dots \alpha_m)$ is a (T, x) -good 2-placement. And if x is a bad vertex in T' , then T is isomorphic to T_D in Fig. 5, and so, by Lemma 3.11, there exists a (T, x) -good 2-placement. Otherwise, suppose that the set of leaves $\{\alpha_1, \dots, \alpha_m\}$ doesn't exist in T . Since T is not a path, neither x nor any of its neighbors is the father of at least two leaves and the set of leaves having the same father doesn't exist, then there exists $y \in N(x)$ such that $T_{(x,y)}$ is a non star tree. Whenever x is a bad vertex in $T_{(x,y)}$, let x_1 and x_2 be the neighbors of x in $T_{(x,y)}$ and y_1 and y_2 be that of x_1 and x_2 , respectively. If there exists $y \in N(x)$ such that $T_{(x,y)}$ and $T_{(y,x)}$ are non star trees, then let σ_y be a $(T_{(y,x)}, y)$ -good 2-placement if y is not a bad vertex in $T_{(y,x)}$, and if it is, then let σ_y be a $(T_{(y,x)}, y')$ -good 2-placement, where y' is a neighbor of y in $T_{(y,x)}$. If x is not a bad vertex in $T_{(x,y)}$, then let σ_x be a $(T_{(x,y)}, x)$ -good 2-placement. The 2-placement σ defined as follows:

$$\sigma(v) = \begin{cases} \sigma_x(v) & \text{if } v \in V(T_{(x,y)}) \\ \sigma_y(v) & \text{if } v \in V(T_{(y,x)}) \end{cases}$$

is a (T, x) -good 2-placement. And if x is a bad vertex in $T_{(x,y)}$, then let T' be the connected component containing x in $T - \{xx_1, xx_2\}$ and let σ_x be a (T', x) -good 2-placement. Thus $\sigma = \sigma_x(x_2 y_2 x_1 y_1)$ is a (T, x) -good 2-placement. Otherwise, if there exists $y \in N(x)$ such that $T_{(x,y)}$ is a non star tree and $T_{(y,x)}$ is a path of length zero, then if x is not a bad vertex in $T_{(x,y)}$, let σ_x be a $(T_{(x,y)}, x)$ -good 2-placement. The 2-placement σ defined such that:

$$\sigma(v) = \begin{cases} \sigma_x(v) & \text{if } v \in V(T_{(x,y)}) - \{x\} \\ y & \text{if } v = x \\ \sigma_x(x) & \text{if } v = y \end{cases}$$

is a (T, x) -good 2-placement. And if x is a bad vertex in $T_{(x,y)}$ then $\sigma = (x y x_1 y_1 x_2 y_2)$ is a (T, x) -good 2-placement. Else, if there exists $y \in N(x)$ such that $T_{(x,y)}$ is a non star tree and $T_{(y,x)}$ is a path of length two, then if x is not a bad vertex in $T_{(x,y)}$, let σ_x be a $(T_{(x,y)}, x)$ -good 2-placement and let $T_{(y,x)} = yzw$. The 2-placement σ defined such that:

$$\sigma(v) = \begin{cases} \sigma_x(v) & \text{if } v \in V(T_{(x,y)}) - \{x\} \\ y & \text{if } v = x \\ \sigma_x(x) & \text{if } v = z \\ w & \text{if } v = y \\ z & \text{if } v = w \end{cases}$$

is a (T, x) -good 2-placement. And if x is a bad vertex in $T_{(x,y)}$, then $\sigma = (x \ y \ w \ z)(x_1 \ y_1 \ x_2 \ y_2)$ is a (T, x) -good 2-placement. Else, there exists $y \in N(x)$ such that $T_{(x,y)}$ is a non star tree and $T_{(y,x)}$ is a path of length one. If $d(x) = 2$, then let $N(x) = \{y_1, y_2\}$. Suppose that $T_{(y_1,x)} = y_1 x_1$ and $T_{(x,y_1)}$ is a non star tree. Let $\{a_1, \dots, a_m\}$, $m \geq 1$, be the neighbors of y_2 distinct from x . If y_2 has a non neighbor F -tree, then suppose that $T_{(a_1,y_2)}$ is that tree and let σ_{a_1} be a $(T_{(a_1,y_2)}, a_1)$ -good 2-placement if a_1 is not a bad vertex in $T_{(a_1,y_2)}$, and if it is, then let σ_{a_1} be a $(T_{(a_1,y_2)}, b)$ -good 2-placement, where b is a neighbor of a_1 distinct from y_2 . If x is not a bad vertex in $T_{(y_2,a_1)}$, then let σ_x be a $(T_{(y_2,a_1)}, x)$ -good 2-placement. Finally, the 2-placement σ defined as follows:

$$\sigma(v) = \begin{cases} \sigma_x(v) & \text{if } v \in V(T_{(y_2,a_1)}) \\ \sigma_{a_1}(v) & \text{if } v \in V(T_{(a_1,y_2)}) \end{cases}$$

is a (T, x) -good 2-placement. If x is a bad vertex in $T_{(y_2,a_1)}$, then y_2 has only two neighbors distinct from x such that $T_{(a_2,y_2)}$ is the vertex a_2 . Let $\{b_1, \dots, b_l\}$, $l \geq 1$, be the neighbors of a_1 distinct from y_2 , T' be the connected component containing x in $T - \{a_1 b_i; i = 1, \dots, l\}$ and let $\sigma' = (x_1 \ x \ y_2 \ y_1)(a_1 \ a_2)$. Whenever $T_{(b_i,a_1)}$, $1 \leq i \leq m$, is not a neighbor F -tree of a_1 , let σ_i be a $(T_{(b_i,a_1)}, b_i)$ -good 2-placement if b_i is not a bad vertex in $T_{(b_i,a_1)}$, and if it is, then let σ_i be a $(T_{(b_i,a_1)}, w_i)$ -good 2-placement, where w_i is a neighbor of b_i in $T_{(b_i,a_1)}$. If all the neighbor trees of a_1 are non neighbor F -trees, then the 2-placement σ defined as follows:

$$\sigma(v) = \begin{cases} \sigma'(v) & \text{if } v \in V(T') \\ \sigma_i(v) & \text{if } v \in V(T_{(b_i,a_1)}) \text{ for } i = 1, \dots, l \end{cases}$$

is a (T, x) -good 2-placement. If a_1 has at least two neighbor F -trees, then suppose that $T_{(b_i,a_1)}$ is a neighbor F -tree of a_1 for $i = 1, \dots, p$, where $2 \leq p \leq l$. By Corollary 3.1, there exists a (T'', x) -good 2-placement σ'' such that $\sigma''(v) = \sigma'(v)$ for every v of T' , where T'' is the connected component containing x in $T - \{a_1 b_i; i = p+1, \dots, l\}$. If $T'' = T$, then σ'' is a (T, x) -good 2-placement. Otherwise, a (T, x) -good 2-placement σ is defined as follows:

$$\sigma(v) = \begin{cases} \sigma''(v) & \text{if } v \in V(T'') \\ \sigma_i(v) & \text{if } v \in V(T_{(b_i,a_1)}) \text{ for } i = p+1, \dots, l \end{cases}$$

Finally, if a_1 has a unique neighbor F -tree, then suppose that $T_{(b_1,a_1)}$ is that tree. Let T' be the connected component containing x in $T - \{a_1 b_i; i = 2, \dots, l\}$, then T' is isomorphic to one of the trees, T_E , T_F or T_G , in Fig. 5, and so, by Lemma 3.11, there exists a (T, x) -good 2-placement σ' such that $\text{dist}(a_1, \sigma'(a_1)) \leq 2$. If $l = 1$, then σ' is a (T, x) -good 2-placement. Else, a (T, x) -good 2-placement σ is defined as follows:

$$\sigma(v) = \begin{cases} \sigma'(v) & \text{if } v \in V(T') \\ \sigma_i(v) & \text{if } v \in V(T_{(b_i,a_1)}) \text{ for } i = 2, \dots, l \end{cases}$$

Now, suppose that all the neighbor trees of y_2 are neighbor F -trees, then $d(y_2) \geq 3$ since T is not a path. Let T_0 be the connected component containing x in $T - \{y_2 a_i; i = 1, \dots, m\}$ and $\sigma_0 = (x \ y_2 \ y_1 \ x_1)$. Then, by Corollary 3.1, there exists a (T, x) -good 2-placement since $\text{dist}(y_2, \sigma_0(y_2)) = 2$ and all the neighbor trees of y_2 are neighbor F -trees. Finally, if $d(x) > 2$,

then each neighbor tree of x is a path of length one. Let $N(x) = \{y_1, \dots, y_r\}$, $r > 2$, and let $T_{(y_i, x)} = y_i x_i$, for $i = 1, \dots, r$. If $d(x) > 4$, then let T' be the connected component containing x in $T - \{xy_i; i = 4, \dots, r\}$. Since T' is a non star tree and x is not a bad vertex in T' , then there exists a (T, x) -good 2-placement. Since $T_{(y_i, x)}$ are neighbor F -trees of x for $i = 4, \dots, r$, then, by Corollary 3.1, there exists a (T', x) -good 2-placement. Otherwise, that is, $d(x) < 5$, then $\sigma = (x y_3 y_2 x_2 x_3 y_1 x_1)$ is a (T, x) -good 2-placement if $d(x) = 3$, and $\sigma' = (x y_3 y_2 x_2 x_3 y_4 x_4 y_1 x_1)$ is a (T, x) -good 2-placement if $d(x) = 4$. \square

Proof of Corollary 1.4.

Let $T' = T - \{\alpha_1, \dots, \alpha_{m_T}\}$, where $\{\alpha_1, \dots, \alpha_{m_T}\}$ is the maximal set of leaves that can be removed from T in such a way that the obtained tree is a non star one. Since T' is a non star tree, then there exists a (T', x) -good 2-placement, say σ_x , where x is any non bad vertex of T' . We define a packing σ of T into T^5 as follows:

$$\sigma(v) = \begin{cases} \sigma_x(v) & \text{if } v \in V(T') \\ \alpha_i & \text{if } v = \alpha_i \text{ for } i = 1, \dots, m_T \end{cases}$$

Label α_i by i for $i = 1, \dots, m_T$ and label all the vertices in T' by $m_T + 1$. Hence, we obtain an $(m_T + 1)$ -labeled packing of T into T^5 , and so $w^5(T) \geq m_T + 1$. \square

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